1

1996

Solution | *Scoring Scale*

Points

(a) $x = -2$

$f'(x)$ changes from positive to negative at $x = -2$, or

f is increasing to the left of $x = -2$ and decreasing to the right of $x = -2$.

$2\begin{cases} 1 : x = -2 \\ 1 : \text{reason} \end{cases}$

(b) $x = 4$

$f'(x)$ changes from negative to positive at $x = 4$, or

f is decreasing to the left of $x = 4$ and increasing to the right of $x = 4$.

$2\begin{cases} 1 : x = 4 \\ 1 : \text{reason} \end{cases}$

(c) $(-1, 1)$ and $(3, 5)$

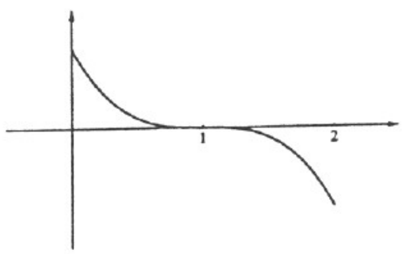
f' is increasing on these intervals.

$3\begin{cases} 1 : (-1,1) \\ 1 : (3,5) \\ 1 : \text{reason using } f' \end{cases}$

max 1/3 for one incorrect interval

0/3 for two incorrect intervals

(d)



$2\begin{cases} <-1> \; f(1) \neq 0 \\ <-1> \; \text{not decreasing} \\ <-1> \; \text{incorrect concavity} \\ <-1> \; f'(1) \neq 0 \end{cases}$

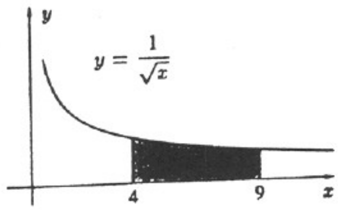
2

Solution

Scoring Scale

Points

(a) $\displaystyle\int_4^9 \frac{dx}{\sqrt{x}} = 2$



$$3\begin{cases} 1 : \text{limits} \\ 1 : \text{integrand} \\ 1 : \text{answer} \\ \quad 0/1 \text{ if integrand is incorrect} \end{cases}$$

(b) $\displaystyle\int_4^k \frac{dx}{\sqrt{x}} = 1$

$$2\sqrt{x}\Big|_4^k = 1$$

$$2\sqrt{k} - 2\sqrt{4} = 1$$

$$k = \frac{25}{4}$$

$$\left(\text{or } \int_k^9 \frac{dx}{\sqrt{x}} = 1 \text{ or } \int_4^k \frac{dx}{\sqrt{x}} = \int_k^9 \frac{dx}{\sqrt{x}}\right)$$

$$3\begin{cases} 1 : \displaystyle\int_4^k \frac{dx}{\sqrt{x}} \text{ or } \int_k^9 \frac{dx}{\sqrt{x}} \\ 1 : \text{equation involving the two halves of } R \\ 1 : \text{answer} \\ \quad 0/1 \text{ if answer from equation not} \\ \quad \text{involving relevant areas} \end{cases}$$

(c) $\displaystyle\text{volume} = \int_4^9 \left(\frac{1}{\sqrt{x}}\right)^2 dx$

$$= \int_4^9 \frac{dx}{x} = \ln x\Big|_4^9 = \ln\frac{9}{4}$$

(or 0.811)

$$3\begin{cases} 1 : \text{limits} \\ 1 : \text{integrand} \\ 1 : \text{answer} \\ \quad 0/1 \text{ if integrand is incorrect} \end{cases}$$

3. The
in b

(a)

(b)

(c)

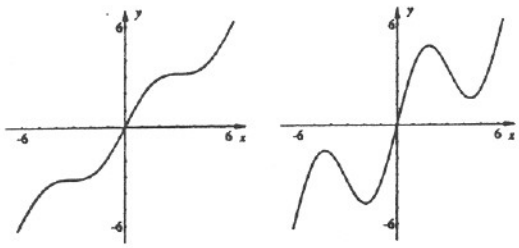
(d)

Solution	Points	*Scoring Scale*
(a) $S(t) = Ce^{kt}$ $S(0) = 6 \implies C = 6$ $S(5) = 12 \implies 12 = 6e^{5k}$ $2 = e^{5k}$ $k = \frac{\ln 2}{5}$ (0.138 or 0.139)	**3**	$\left\{\begin{array}{l} 1: C = 6 \\ 1: 12 = 6e^{5k} \\ 1: k = \frac{\ln 2}{5} \end{array}\right.$
(b) average rate $= \frac{1}{13-3}\int_3^{13} 6e^{\frac{\ln 2}{5}t}\,dt$ $= \frac{3}{\ln 2}\left[e^{2.6\ln 2} - e^{0.6\ln 2}\right]$ billion gal/yr (19.680 billion gal/yr)	**3**	$\left\{\begin{array}{l} 1: \text{uses } [3,13] \text{ and divides by } 13-3 \\ 1: \text{integrand} \\ 1: \text{answer with units} \\ \quad 0/1 \text{ if not } \frac{1}{b-a}\int_a^b S(t)\,dt \end{array}\right.$
(c) $\int_5^7 S(t)\,dt$ $\doteq \frac{1}{4}[S(5) + 2S(5.5) + 2S(6) + 2S(6.5) + S(7)]$	**1**	$\left\{\begin{array}{l} \text{Trapezoidal rule with } S, \\ n = 4, \text{ interval } [5,7] \end{array}\right.$
(d) This gives the total consumption, in billions of gallons, during the years 1985 and 1986.	**2**	$\left\{\begin{array}{l} 1: \text{total consumption in a time period} \\ 1: \left\{\begin{array}{l} \text{correct time period} \\ \text{liquid measure} \end{array}\right. \end{array}\right.$ 0/2 for "rate of consumption"

Solution | *Scoring Scale*

Points

(a)



2 {
1 : graph of $y = x + \sin x$ (increasing, through origin, cd - cu - cd - cu)
1 : graph of $y = x + 3\sin x$ (inc - dec - inc - dec - inc, through origin, cd - cu - cd - cu)
}

(b) $y' = 1 = 1 + b\cos x$

$b\cos x = 0$

$\cos x = 0$

$y = x + b = x + b\sin x$

$b = b\sin x$

$1 = \sin x$

$x = -\dfrac{3\pi}{2}$ or $\dfrac{\pi}{2}$

4 {
1: derivatives of $x + b$ and $x + b\sin x$
1: sets derivatives equal
1: sets y-values equal
1: answer from merged information
}

max: 0 - 1 - 1 - 0 if specific value(s) for

max: 0 - 0 - 1 - 0 if no derivative

(c) No, because $f'(x) = 1$ (or $f'(x) \neq 0$) at x-coordinates of points of tangency.

1: answer with reason

(d) $f''(x) = -b\sin x = 0$

$\sin x = 0$

$f(x) = x + b \cdot 0 = x$
at x-coordinates of any inflection points

2 {
1: sets second derivative equal to 0
1: shows $f(x) = x$
}

0/2 if specific value(s) for b

Solution

(a) volume $= V = \pi \int_0^9 \frac{25}{3}\sqrt{y}\,dy = 150\pi \text{ ft}^3$

(or 471.238 ft^3 or 471.239 ft^3)

$$V = 2\pi \int_0^5 x\left(9 - \frac{9}{625}x^4\right)\,dx$$

(b) time $= \frac{\text{volume}}{\text{rate}} = \frac{150\pi}{8}$

therefore, 59 minutes

(c) $V = \pi \int_0^h \frac{25}{3}\sqrt{y}\,dy$

$$\frac{dV}{dt} = \frac{25}{3}\pi\sqrt{h}\,\frac{dh}{dt}$$

$$\frac{dV}{dt} = 8$$

when $h = 4$, $8 = \frac{25}{3}\pi(2)\frac{dh}{dt}$

$$\frac{dh}{dt} = \frac{12}{25\pi} \text{ ft/min}$$

(or 0.152 ft/min or 0.153 ft/min)

Scoring Scale

Points

$$3 \begin{cases} 2: \text{integral} \begin{cases} 1: \text{limits and } \pi/2\pi \\ \quad \pi\int_0^9 (\ )\,dy, \quad 2\pi\int_0^5 (\ )\,dx \\ 1: \text{integrand} \end{cases} \\ 1: \text{answer with units} \end{cases}$$

not eligible for answer point unless first two points are earned.

1: integer answer

$$5 \begin{cases} 1: \text{volume as definite integral using } h \\ 2: \text{finding } \frac{dV}{dt} \begin{cases} 1: \frac{dV}{dh} \\ 1: \frac{dV}{dt} = \frac{dV}{dh}\cdot\frac{dh}{dt} \end{cases} \\ 1: \frac{dV}{dt} = 8 \\ 1: \text{answer with units} \end{cases}$$

if V is linear, max 2/5 (0 - 0 - 1 - 1 - 0)
if V is constant, max 1/5 (0 - 0 - 0 - 1 - 0)

Solution | *Scoring Scale*

Points

(a) let Q be $\left(a, a - \dfrac{a^2}{500}\right)$.

$$\left[\begin{array}{l} \dfrac{dy}{dx} = 1 - \dfrac{x}{250} \\ \text{setting slopes equal:} \\ 1 - \dfrac{a}{250} = \dfrac{\left(a - \dfrac{a^2}{500}\right) - 20}{a} \\ a = 100 \end{array}\right.$$

or

$$\left[\begin{array}{l} \dfrac{dy}{dx} = 1 - \dfrac{x}{250} \\ \text{equation for } \ell : y = \left(1 - \dfrac{a}{250}\right) x + 20 \\ \text{setting } y\text{-values equal:} \\ \left(1 - \dfrac{a}{250}\right) a + 20 = a - \dfrac{a^2}{500} \\ a = 100 \end{array}\right.$$

4
- 1 : slope of tangent line from parabola
- 1 : uses the condition that $\left(a, a - \dfrac{a^2}{500}\right)$ is on line ℓ
- 1 : uses the condition that slopes are equal at Q
- 1 : answer
  0/1 if student is solving an irrelevant equation

(b) $y = \dfrac{3}{5}x + 20$

2
- 1 : slope
  0/1 if $m \leq 0$
- 1 : equation

(c) height of hill at $x = 250$:

$y = 250 - \dfrac{250^2}{500} = 125$ feet

height of line at $x = 250$:

$y = \dfrac{3}{5}(250) + 20 = 170$ feet

Yes, the spotlight hits the tree since the height of the line is less than the height of the hill + tree which is 175 feet.

3
- 1 : height of hill
- 1 : height of line
  0/1 if height < 0
- 1 : answer with analysis